A Simplified Formula of Surge Characteristics of a Long Grounding Conductor

S.Sekioka[1], M.I.Lorentzou[2], and N.D.Hatziargyriou[2]

(1) Kansai Tech Corporation, 3-1-176 Fukuzaki, Minato-ku, Osaka, 552-0013, Japan (e-mail: sho.sekioka@tech.email.ne.jp), (2) Department of Electrical and Computer Engineering, National Technical University of Athens, Athens, 157-73, Greece (e-mail: lorentz@power.ece.ntua.gr)

Abstract **– This paper proposes a formula to estimate surge characteristics of a horizontal long grounding conductor. The formula is derived in time domain on the basis of a lattice diagram method, and is very simple to do rough estimation. This paper describes propagation characteristics of voltages on the grounding conductor using the proposed formula. Accuracy of the formula and an effective length of the long grounding conductor are discussed.**

Keywords – Long grounding conductor, Surge characteristic, Effective length, and Lattice diagram method

I. INTRODUCTION

A long grounding conductor is frequently used to obtain low grounding resistance. Surge characteristics of the grounding conductor should be considered for steep-front currents. Voltage propagating on a distributed-parameter line is generally expressed as follows:

$$
\begin{aligned}
V &= V_1 e^{-\Gamma x} + V_2 e^{\Gamma x} \\
\Gamma &= \sqrt{ZY} \\
Z &= R_d + jwL_d \\
Y &= G_d + jwC_d
\end{aligned}
\qquad (1)
$$

where Z: line series impedance per unit length, Y:, line shunt admittance per unit length, R_d: series resistance per unit length, L_d: series inductance per unit length, G_d: shunt conductance per unit length, C_d: shunt capacitance per unit length, x: coordinates along the conductor, w: angular frequency. The shunt conductance of an overhead conductor is usually ignored. Lattice diagram method [1] and Dommel's method [2] are frequently used for surge analysis, and adopt eq. (1) of a loss-less line with R_d=G_d=0. On the other hand, the conductance of a buried bare conductor in the ground plays an important role. It is difficult to represent the surge characteristics of a long grounding conductor in time domain even assuming R_d=0 because the propagation constant G is a function of frequency due to the presence of the shunt conductance. Numerical computation is, therefore, mainly used to simulate the surge characteristics of the long grounding conductor [3-5]. However, it is not convenient for rough estimation, and the simulation does not directly show a physical meaning of the surge characteristics. Thus, a simplified formula is required to estimate measurement and simulation results.

The authors derive an analytical formula of sending-end voltage on the long grounding conductor in time domain based on the lattice diagram method. The proposed formula is simple and convenient. Frequency dependence of the conductor constants is sometimes considered. This dependence should be treated in frequency domain, and this paper takes the frequency dependence into no consideration. This paper describes propagation characteristics of voltage and an effective length of the long grounding conductor using the proposed formula.

II. EQUIVALENT CIRCUIT OF LONG GROUNDING CONDUCTOR

Fig. 1 illustrates a part of an equivalent circuit of a long grounding conductor. The equivalent circuit is composed of the shunt conductance due to finite soil resistivity and the loss-less line. The equivalent circuit shows sufficient accuracy as described in Appendix A. Coefficients *A*, *B* and *A'* in Fig. 1 are given by:

$$
\begin{aligned}
A &= \frac{2Z_0 // R}{Z_0 // R + Z_0} = \frac{2R}{Z_0 + 2R} \\
B &= A - 1 = \frac{-Z_0}{Z_0 + 2R} \\
A' &= A
\end{aligned}
\qquad (2)
$$

where R=$(G\Delta x)^{-1}$, A: refraction coefficient from k to k+1, B: reflection coefficient from k to k+1, A': refraction coefficient from k+1 to k, Z_0: surge impedance of the loss-less line, v: surge velocity on the loss-less line, Dx: elementary length of the line. The coefficients are real number. Therefore, voltages on the grounding conductor can be expressed in time domain using the lattice diagram method.

III. DERIVATION OF SIMPLIFIED FORMULA

A. Assumptions to Derive Formula

The following assumptions are adopted to derive a simplified expression.

(a) Series resistance is ignored.

(b) Conductor constants are independent of frequency, and soil resistivity is homogenously distributed.

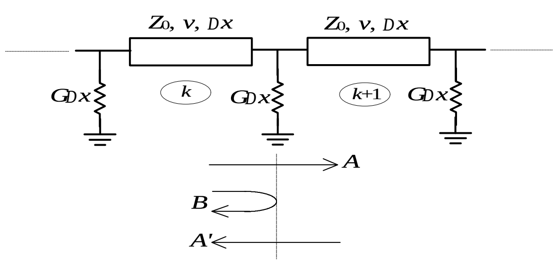


Fig. 1 An equivalent circuit of a long grounding conductor

(c) Voltages reflected more than three times on the long grounding conductor are ignored.

(d) π−circuit representation composed of a series inductance and a shunt capacitance with infinitesimal length is equivalent to a loss-less line.

B. Basic Idea

The lattice diagram method [1] is applicable to a non-uniform line such as tapered line [6] and nonparallel line [7]. The surge impedance of the non-uniform line varies along the line, and voltages reflected on the line due to the discontinuity of the surge impedance are generated. Sending end voltage is given by the sum of applied voltage and the reflected voltages. An analytical formula of the voltage on the non-uniform line can be obtained as $\Delta x \to 0$.

Considering the reflection voltages within length *x*, where *x* is the distance from the sending end, the number of segments *n* is denoted by:

$$n = \frac{x}{\Delta x} \qquad (3).$$

The coefficients *A* and *B* can be rewritten as follows:

$$A = \frac{1}{1+q\Delta x}$$
$$B = \frac{-q\Delta x}{1+q\Delta x} \qquad (4)$$
$$q = \frac{1}{2} Z_0 G$$

The surge impedance of a horizontal long grounding conductor is independent of location, and the shunt conductance is distributed along the conductor. Consequently, voltages reflected and refracted on the grounding conductor appear at each nodes of the conductor as shown in Fig. 1. The reflection and refraction coefficients are real number and constant. Coefficient of voltage with *a*th order reflection and *b*th order refraction is denoted by $A^a B^b A'^a$.

Fig. 2 illustrates a lattice diagram along a long grounding conductor, where P_s and Q_s are refraction and reflection coefficients from the grounding conductor to the voltage source, respectively. For simplicity, only *B*, $Q_s B^2$ and B^3 components are drawn on the figure. Sending-end voltage is observed at point *K*.

C. 1st Order Reflection Voltage on the Conductor

Voltage V_b' reflected once on the grounding conductor is given by:

$$V_b' = EB\sum_{i=1}^{n} A^{i-1}A'^{i-1} = EB\sum_{i=1}^{n} A^{2(i-1)}$$
$$= EB\frac{1-A^{2n}}{1-A^2} \qquad (5)$$

where *E*: original sending end voltage with step waveform. The influence of the first shunt conductance on the sending end voltage can be ignored. The original sending end voltage *E* of the circuit in Fig. 2 is given by:

$$E = \frac{Z_0}{Z_0 + R_s} E_0 \qquad (6).$$

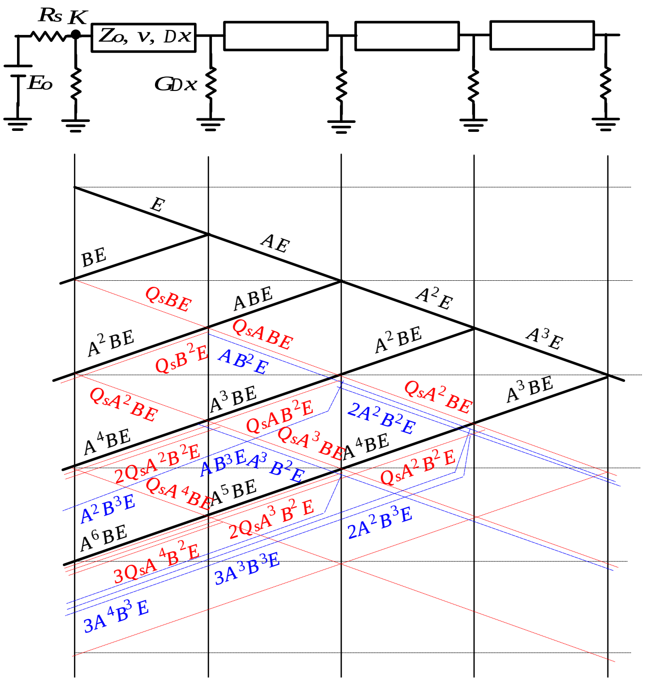


Fig. 2 A lattice diagram along a long grounding conductor.

Substituting the following relations (see Appendix *B*)

$$\lim_{\Delta x \to 0} A^{2n} = e^{-2qx} \qquad (7)$$

$$\lim_{\Delta x \to 0} \frac{B}{1-A^2} = -\frac{1}{2} \qquad (8)$$

into eq. (5), the reflection voltage V_b' as a limit

$$\lim_{\Delta x \to 0} V_b' = -\frac{1}{2}E\left(1-e^{-2qx}\right) \qquad (9)$$

is derived. *x* is the location of the head of the traveling voltage, and

$$x = \frac{1}{2}vt \qquad (10).$$

D. 3rd Order Reflection Voltage on the Conductor

Voltages reflected three times or more on the conductor are sometimes necessary to get higher accuracy. From the assumption (c), only third-order *B* component V_b” is considered here. Number N_i of paths of the component $B^3A^{2(i-1)}$ is given by:

$$N_i = \frac{1}{2}(i-1)i \qquad (11).$$

Accordingly, the voltage of the third-order *B* component is given by:

$$V_b'' = EB^3\sum_{i=1}^{n-1} N_i A^{2(i-1)}$$
$$= E\frac{B}{1-A^2}\left[\frac{B}{1-A^2}\left\{\frac{B}{1-A^2}\left(A^2 - A^{2(n-1)}\right) - (n-2)BA^{2(n-1)}\right\} - \frac{1}{2}(n-2)(n-1)B^2A^{2(n-1)}\right] \qquad (12).$$

From eqs. (7), (8), (12) and

$$\lim_{\Delta x \to 0} nB = -qx \qquad (13),$$

$$\lim_{\Delta x \to 0} V_b" = E\left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right)\left\{\left(-\frac{1}{2}\right)\left(1-e^{-2qx}\right) + (-qx)e^{-2qx}\right\} - \frac{1}{2}(-qx)^2 e^{-2qx}\right] = -\frac{1}{4}E\left[\frac{1}{2}\left(1-e^{-2qx}\right) - qx(1+qx)e^{-2qx}\right] \quad (14)$$

is obtained.

E. Voltage Reflected at the Sending End

Voltage, which is reflected on the grounding conductor, and then reflected at the sending end, is reflected again on the grounding conductor. This voltage V_a is observed at the sending end, and is given by:

$$V_a = EQ_s B^2 \sum_{i=1}^{n-1} M_i A^{2(i-1)} = EQ_s \frac{B}{1-A^2}\left[\frac{B}{1-A^2}\left\{1-A^{2(n-1)}\right\} - (n-1)BA^{2(n-1)}\right] \quad (15)$$

where $M_i = i$. Substituting eqs. (7), (8) and (13) into eq. (15),

$$\lim_{\Delta x \to 0} V_a = \frac{1}{2}EQ_s\left\{\frac{1}{2}\left(1-e^{-2qx}\right) - qxe^{-2qx}\right\} \quad (16)$$

is obtained.

The sending end voltage, until an influence of the receiving end of the conductor appears, is calculated considering a boundary condition at the sending end as follows:

$$V = E + P_s(V_b' + V_b" + V_a) \quad (17).$$

Eqs. (9), (14) and (16) are very simple, and it is very convenient to carry out rough estimation of surge characteristics of a long grounding conductor.

IV. Verification of the Proposed Formula

This chapter verifies accuracy of the proposed formula for the circuit illustrated in Fig. 3. Table I shows test cases for the verification, where l=100m, v=100m/µs, which equals one-third speed of light in free space, and E_0=1p.u. Steady-state grounding resistance R_0 of the long grounding conductor is given by:

$$R_0 = (Gl)^{-1} \quad (18).$$

Ref. [8] suggests on the basis of experimental results that surge impedance of a horizontal long grounding conductor is not dependent on its length and soil resistivity. Here the surge impedance of about 100Ω is used.

Fig. 4 shows calculated waveforms of sending end voltage V_s by the proposed formula (17) with solid line and exact solution with dotted line. Maximum observation time is twice traveling time of the grounding conductor. The exact solution is obtained by calculating the following equation using a numerical Laplace transform [9].

$$V_s = L^{-1}\left\{\frac{Z_{0f}\coth(\Gamma_f l)}{Rs + Z_{0f}\coth(\Gamma_f l)}E\right\} \quad (19)$$

where $Z_{0f} = \sqrt{sL/(G+sC)}$, $\Gamma_f = \sqrt{sL(G+sC)}$, s: Laplace operator, l: length of the grounding conductor

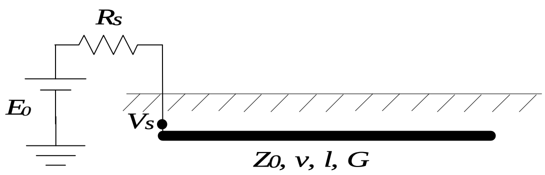


Fig. 3 Test circuit for verification of the proposed formula

Table I Test cases for verification

Case	Z_0 [Ω]	R_0 [Ω]	R_s [Ω]
1	100	200	100
2	100	100	100
3	100	50	100
4	100	10	100
5	100	50	200
6	100	50	50

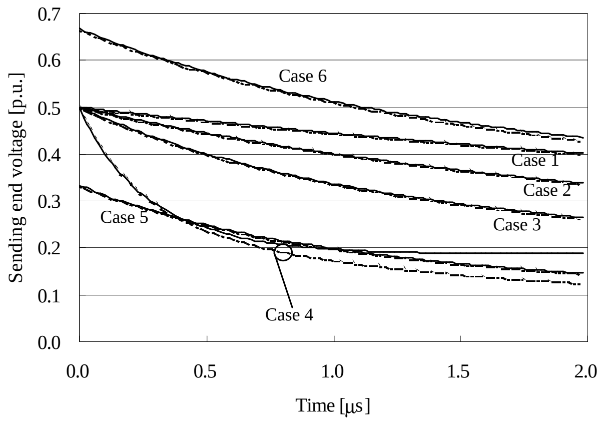


Fig. 4 Comparison of voltage waveforms by the proposed formula with exact solution

It is clear from Fig. 4 that the proposed formula has a sufficient accuracy except for case 4. More than third-order B component might be needed to get higher accuracy.

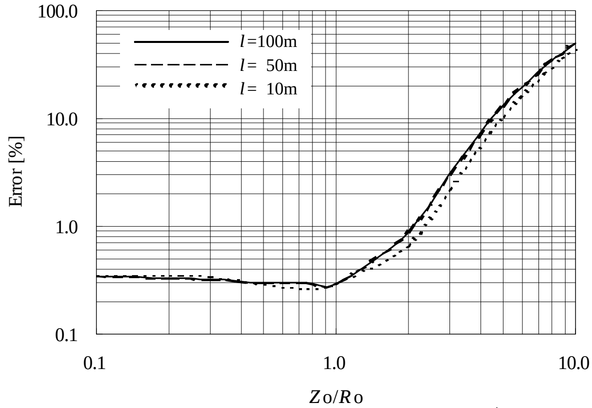
Let us investigate an influence of the surge impedance and the grounding resistance of the grounding conductor on accuracy of the proposed formula. Fig. 5 shows an error, which is estimated at x=l, as a function of Z_o/R_o and R_o.

As shown in Fig. 5(a), the proposed formula has high accuracy for Z_o/R_o<1. The error increases as Z_o/R_o becomes larger, and is not dependent on the conductor length. Fig. 5(b) shows that the error is large for low steady-state grounding resistance. $qx(1+qx)$ can be approximated for $2qx$<<1 as follows:

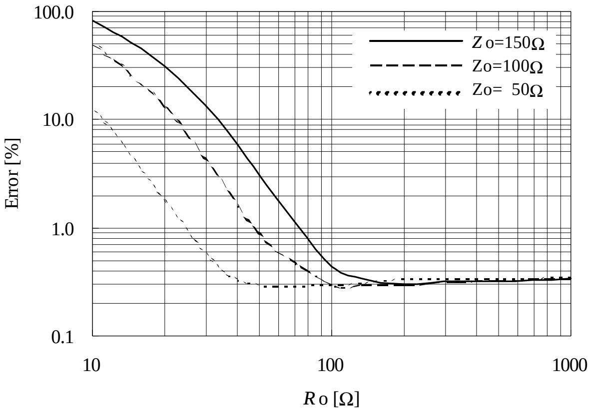
$$qx(1+qx) = \frac{1}{2}\left\{1 + (2qx) + \frac{1}{2}(2qx)^2 - 1\right\} \approx \frac{1}{2}\left\{\exp(2qx) - 1\right\} \quad (20).$$

Consequently V_b" almost equals 0 in case of $2qx$<<1. Thus, high-order B components are negligible for small $2qx$, and eqs. (9) and (14) are well approximated. $2ql$ is rewritten as:

$$2ql = Z_0 Gl = \frac{Z_0}{R_0} \quad (21).$$



(a) Z_o/R_o (Z_o=100Ω, v=100m/μs, R_s=Z_o)



(b) R_o (v=100m/μs, l=100m, R_s=Z_o)

Fig.5 Accuracy of the proposed formula

Accordingly, the proposed formula has sufficient accuracy for Z_o/R_o<1, and the error becomes larger for large Z_o/R_o. Higher-order B components in order to make error small are necessary as Z_o/R_o is larger.

V. DISCUSSIONS

A. Surge Characteristic using Conductor Constants by Sunde's Formulas

Sunde derived formulas of constants of a horizontal long grounding conductor.

Inductance L: $L = \frac{m_0}{2p} W$

Capacitance C: $C = 2pe_r e_0 W^{-1}$

Conductance G: $G = \frac{p}{r} W^{-1}$ (22)

where $W = \ln \frac{2l}{\sqrt{2rd}} - 1$, ρ: soil resitivity, e_r: soil relative permittivity, d: burial depth, r: conductor radius. The surge impedanceZ_0 and the velocity v of the loss-less line of the equivalent circuit shown in Fig. 1 are given by:

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{60W}{\sqrt{e_r}} \quad (23)$$

$$v = \frac{1}{\sqrt{LC}} = \frac{v_0}{\sqrt{e_r}} \quad (24)$$

where v_o: speed of light in free space. Substituting the Sunde's formulas into q,

$$2qx = \frac{30p}{r\sqrt{e_r}} \frac{v_0}{\sqrt{e_r}} t = \frac{30p}{re_r} v_0 t \quad (25)$$

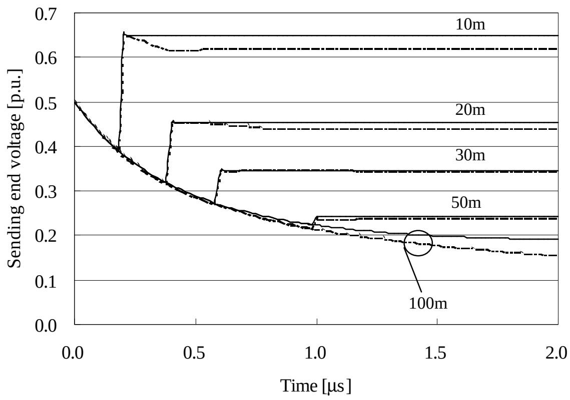
is obtained. Eq. (25) indicates that wave deformation of the sending-end voltage is independent of the configuration and dimension of the grounding conductor. On the other hand, the surge impedance depends on them.

Fig. 6 shows sending-end voltage waveforms on a grounding conductor with r=5mm and d=1m for various conductor lengths. A single grounding conductor is usually open-circuited at the receiving end. Voltage reflected at the receiving end is observed at the sending end for t>2T, where T is traveling time of the grounding conductor. Fig. 6 considers only the reflection voltage with zero-order B component for simplicity. The sending end voltage for t>2T is given by the reflection voltage plus V_b', V_b" and V_a at t=2T. The reflection voltage V_r' is given by:

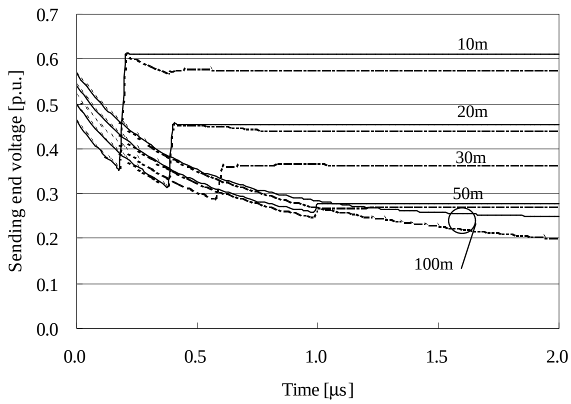
$$V_r' = EQ_r A^{2n} \quad (26)$$

where Q_r: reflection coefficient at the receiving end. V_r' as a limit is expressed by:

$$\lim_{\Delta x \to 0} V_r' = EQ_r e^{-2ql} \quad (27).$$



(a) R_s=Z_0



(b) R_s=100Ω

Fig.6 Voltage waveforms at sending end for various conductor lengths (ρ=1000Ωm, the proposed formula: solid line, exact solution: broken line)

Line constants of an overhead conductor are independent of the conductor length, and propagation characteristics on the conductor for various lengths are same until an influence of the receiving end appears. The grounding conductor must satisfy the characteristic. Experimental results on a horizontal grounding conductor verifies this characteristic [11]. Fig. 6(a) shows this characteristic, but the voltage in Fig. 6(b) is dependent on the conductor length. The surge impedance of the grounding conductor using the Sunde's formula is denoted as a function of the conductor length. Considering that the original sending-end voltage is dependent on the surge impedance, the sending-end voltage is affected by the conductor length due to the Sunde's formula. As previously mentioned, the wave deformation of the grounding conductor is independent of the conductor constants. The constants of the Sunde's formula should be chosen carefully.

B. Effective Length

"Effective length" is defined as the length, above which no further reduction of impedance of a grounding conductor is observed [12], and is estimated in frequency and time domains [12-14]. The faster the injected current reaches the peak value, the shorter the effective length becomes. The impulse impedance of the conductor in time domain will be very close to the value of impedance in frequency domain, at a particular frequency of sinusoidal wave. This paper estimates the effective length in frequency domain. Voltage reflected at the receiving end can not be clearly observed for conductor of 100m in Fig. 6. This fact indicates that an observer can not recognize the receiving end for grounding conductor having a certain length. This paper discusses this "characteristic length". When V_r' is much less than V_b at $x=l$, namely $(V_r'/V_b)_{x=l}<\delta$, where d is a small value, the receiving end is not observed. Considering d is very small, the relation can be written as follows:

$$l_{ch} \cong \frac{1}{2q}\ln\left(\frac{5}{8}d\right) \qquad (28).$$

Table II shows the effective length and the characteristic length of a grounding conductor with r=7mm and d=0.5m buried in soil with parameters of r and e_r. δ of 1% is selected. Table II includes fundamental resonance frequency of an open-end line $f_c=1/(4T)$ [15].

From Table II, the characteristic length is roughly equal to the effective length in the fundamental resonance frequency.

VI. CONCLUSIONS

This paper has described a formula of voltage on a long grounding conductor. The formula is denoted in time domain, and is very simple. Calculated results by the formula agree satisfactorily with exact solutions for small Z_o/R_o. The proposed formula makes clear that surge characteristics using Sunde's formula is affected by the conductor length because the surge impedance using the Sunde's formula depends on the length.

REFERENCES

[1] L.V. Bewley, "Traveling Waves on Transmission Systems," John Wiley, New York, 1951.

[2] H.W. Dommel, "Digital computer solution of electromagnetic transients in single- and multi-phase networks," IEEE Trans., Power App. and Syst., Vol. PAS-88 (4), pp. 388-398, 1969.

[3] A.P.Meliopoulos, "Power System Grounding and Transients," Marcel Dekker, Inc., New York, 1988.

[4] F.E. Menter, and L. Grcev, "EMTP-based model for grounding system analysis," IEEE Trans., Power Delivery, Vol. PWRD-9 (4), pp. 1838-1849, 1994.

[5] S. Yamaguchi, M. Inoue, S. Sekioka, T. Sonoda, Y. Kato, N. Nagaoka, and A. Ametani, "A frequency-dependent counterpoise model for a transient analysis," Proc. of ICEE'98, Vol. 1, pp. 753-756, Kyongju, Korea, 1998.

Table II Effective length vs. characteristic length

ε_r	ρ=200Ωm		ρ=600Ωm		ρ=1200Ωm		ρ=1600Ωm		ρ=2500Ωm	
	Frequency	Length	Frequency	Length	Frequency	Length	Frequency	Length	Frequency	Length
4	7.5E5	10	1.4E6	10	8.0E5	10	6.2E5	10	5.4E5	10
	1.4E5	20	6.5E5	20	1.0E6	20	5.9E5	20	5.0E5	20
	2.6E4	50	8.5E4	50	2.2E5	50	3.5E5	50	4.0E5	50
	7.4E3	100	1.9E4	100	4.3E4	100	5.8E4	100	1.3E5	100
	1.9E3	200	5.2E3	200	1.0E4	200	1.4E4	200	1.9E4	200
	9.0E2	300	1.8E3	300	4.8E3	300	4.7E3	300	1.0E4	300
	*3.5E6	*11	*1.2E6	*32	*5.8E5	*65	*4.4E5	*86	*2.8E5	* 135
20	9.0E5	10	5.0E5	10	2.4E5	10	1.4E5	10	1.3E5	10
	1.1E6	20	4.3E5	20	2.2E5	20	1.3E5	20	1.1E5	20
	1.7E5	50	3.7E5	50	2.0E5	50	1.3E5	50	9.6E4	50
	3.3E4	100	1.8E5	100	1.7E5	100	1.2E5	100	8.0E4	100
	9.3E3	200	4.0E4	200	7.3E4	200	1.3E5	200	6.6E4	200
	4.8E3	300	1.3E4	300	2.7E4	300	3.0E4	300	1.4E5	300
	*7.0E5	*24	*2.3E5	*72	*1.2E5	* 145	*8.7E4	* 193	*5.6E4	* 301
80	3.7E5	10	1.2E5	10	6.0E4	10	4.5E4	10	2.7E4	10
	3.4E5	20	1.1E5	20	5.7E4	20	4.2E4	20	2.5E4	20
	3.1E5	50	8.0E4	50	5.2E4	50	3.5E4	50	2.1E4	50
	3.9E5	100	8.0E4	100	5.0E4	100	3.2E4	100	1.9E4	100
	5.0E4	200	9.0E4	200	3.7E4	200	2.9E4	200	1.7E4	200
	2.2E4	300	1.2E5	300	3.2E4	300	2.7E4	300	1.6E4	300
	*1.7E5	*48	*5.8E4	*15	*2.9E4	* 289	*2.2E4	* 385	*1.4E4	* 602

*Characteristic length and fundamental resonance frequency

[6] L.O. Barthold, "An approximate transient solution of the tapered transmission line," AIEE Trans., Power App. and Syst., Vol. PAS-77, pp. 1556-1561, 1958.
[7] K. Kotaki, A. Hashimoto, H. Isa, and S. Sekioka, "Wave propagation characteristics of a lightning surge on an incoming line," Trans. of IEE of Japan, Vol. 113-B (9), pp. 1019-1028, 1993 (in Japanese).
[8] H. Motoyama, and T. Noda, "Experimental study on lightning surge characteristics of a buried bare wire," CRIEPI Report, No. T00062, 2001 (in Japanese).
[9] A. Ametani, "The application of the fast Fourier transform to electrical transients phenomena," Int. J. Elect. Eng. Educ., Vol. 10, pp. 277-281, 1973.
[10] E.D. Sunde, "Earth Conduction Effects in Transmission Systems," *D. Van Nostrand Co.*, New York, 1949.
[11] A. Ametani, M. Nayel, S. Sekioka, and T. Sonoda, "Basic investigation of wave propagation characteristics on an underground naked conductor," Proc. of ICEE'2002, pp. 2141-2146, Jeju, Korea, 2002.
[12] M.I. Lorentzou, and N.D. Hatziargyriou, "Modelling of long grounding conductors using EMTP," Proc. of IPST'99, pp. 346-351, Budapest, Hungary, 1999.
[13] M.I. Lorentzou, "Contribution to modeling and analysis of the transient behavior of grounding systems," PhD Thesis, National Technical University of Athens, November, 2001
[14] L. Grcev, "Improved earthing system design practices for reduction of transient voltages," Paper 36-302, CIGRE Conference, Paris, France, 1998
[15] A. Ametani, "Distributed-parameter circuit theory," *Corona Pub.*, Tokyo, 1990 (in Japanese).

APPENDICES

A. Accuracy of the Equivalent Circuit

A general equivalent circuit of distributed-parameter line is represented by series impedance and shunt admittance. The circuit illustrated in Fig. 1 is constructed by dividing a grounding conductor into elementary segments each of which is represented by a modified π-circuit. In the modified π-circuit, inductance and capacitance elements are considered as a distributed-parameter line in the middle of each segment while the resistance and conductance are considered lumped. The numerical error introduced by this technique is only due to the fact that R and G parameters are considered lumped. This is opposed to the distributed nature of R and G.

Fig. 1 with series resistance can be transformed to Fig. A1 [13]. The valuables are denoted as follows:

$$Z_S^* = \frac{R\cdot\sinh(g_0\Delta x) + 2\cdot Z_0\cdot(\cosh(g_0\Delta x)-1)}{2\cdot\sinh(g_0\Delta x)}\cdot\left(1+\frac{R\cdot\sinh(g_0\Delta x)}{2\cdot Z_0}+\cosh(g_0\Delta x)\right) \quad \text{(A1)}$$

$$Z_p{}^* = \frac{Z_p{}^{**}}{2+GZ_p{}^{**}} \quad \text{(A2)}$$

$$Z_p^{**} = \frac{R\cdot\sinh(g_0\Delta x) + 2\cdot Z_0\cdot(\cosh(g_0\Delta x)+1)}{2\cdot\sinh(g_0\Delta x)} \quad \text{(A3)}$$

where $Z_0 = \sqrt{L_d / C_d}$, $g_0 = jw\sqrt{L_d C_d}$.

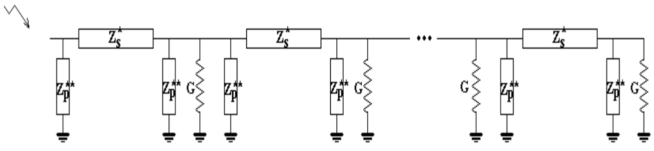


Fig. A1 A transformed equivalent circuit

The current source connected to the end point of the conductor in Fig. A1 "sees" the impedance of the conductor's n-π circuit ladder network as:

$$Z = \frac{a_1\left(b-Z_e^*\right)^n - a_2\left(b+Z_e^*\right)^n}{\left(b+Z_e^*\right)^n - \left(b-Z_e^*\right)^n} \quad \text{(A4)}$$

where $a_1 = -(Z_s^* - Z_e^*)/2$, $a_2 = -(Z_s^* + Z_e^*)/2$, $b = Z_s^* + 2Z_p^*$,

$$Z_e^* = \sqrt{Z_s^{*2} + 4Z_s^* Z_p^*}$$

As n tends to infinity, the limit of the impedance Z can be evaluated:

$$\lim_{n\to\infty} Z = \lim_{n\to\infty}\frac{a_1 b^n - a_2}{1-b^n} \quad \text{(A5)}$$

where $b = \left\{b - \sqrt{Z_s^{*2} + 4Z_s^* Z_p^*}\right\} / \left\{b + \sqrt{Z_s^{*2} + 4Z_s^* Z_p^*}\right\}$

The limit of Z is obtained using the same manner in Ref. [12]:

$$\lim_{n\to\infty} Z = \lim_{n\to\infty}\frac{Z_c e^{-2g_c l} + Z_c}{1-e^{-2g_c l}} = Z_c\cdot\coth(g_c l) \quad \text{(A6)}$$

where $Z_c = \sqrt{(R_d + jwL_d)/(G_d + jwC_d)}$,

$g_c = \sqrt{(R_d + jwL_d)(G_d + jwC_d)}$.

Eq. (A6) derived from Fig. 1 with series resistance coincides with exact input impedance of an open-ended transmission line considering the series resistance and the shunt conductance. Thus, Fig. 1 is an equivalent circuit of exact long grounding conductor.

B. Derivation of eqs. (7) *and* (8)

(i) Eq. (7)

$$\lim_{\Delta x\to 0}\ln A^{2n} = \lim_{\Delta x\to 0}\frac{-2x}{\Delta x}\ln(1+q\Delta x)$$

$$= \lim_{\Delta x\to 0}(-2x)\frac{q}{1+q\Delta x}$$

$$= -2qx$$

$$\therefore \lim_{\Delta x\to 0} A^{2n} = e^{-2qx}$$

(ii) Eq. (8)

$$\frac{B}{1-A^2} = \frac{B}{(1-A)(1+A)} = \frac{-1}{1+A}$$

$$\lim_{\Delta x\to 0} A = 1$$

$$\therefore \lim_{\Delta x\to 0}\frac{B}{1-A^2} = -\frac{1}{2}$$