# A New Method of MODAL Analysis Applied to the Modelling of Power Transmission Lines

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*Abstract* — **The paper presents an entirely new approach to transmission-line modelling resulting in a model structured around natural MODES of oscillation, unlike previous models structured around natural MODES of wave propagation. The principal advantage is that the new model converts easily from the frequency domain to the time domain to give a highly efficient structure suited to economical EMTP implementation.**

**Keywords :** Transmission Lines, Modal Analysis, EMTP, Transients, Frequency Dependence

## I. INTRODUCTION

Computer modelling of power transmission lines is generally based on the solution of the well-known multiconductor Telegraphers' equations :

$$\frac{dV(x)}{dx} = -Z(x)I(x) \tag{1}$$

$$\frac{dI(x)}{dx} = -Y(x)V(x) \tag{2}$$

where $V(x)$ and $I(x)$ are vectors of dimension equal to the number of conductors. Equations (1) and (2) are understood to be in the frequency domain where they are able to take account of well-known frequency-dependent effects, notably owing to the nature of the ground return path.

Normally it is assumed that the transmission line is longitudinally homogeneous (at least over defined lengths) so that $Z$ and $Y$ are independent of the distance parameter $x$. In particular, this allows (1) to be differentiated as

$$\frac{d^2V(x)}{dx^2} = -Z\frac{dI(x)}{dx} \tag{3}$$

Direct inward substitution, using (2) to eliminate $I(x)$, then gives the equation

$$\frac{d^2V(x)}{dx^2} = ZY\,V(x) \tag{4}$$

whose solution, as is well known, corresponds to wave propagation which can be decomposed into natural modes by diagonalising $ZY$ [1].

This particular procedure, and consequent model structure, is so well known that it appears that many researchers may have assumed that this is the only possible solution procedure. The present paper describes a completely different approach which leads to an entirely different model structure. Both methods necessarily yield the same results when applied in the frequency domain. The advantage of the new approach is that it *structures* the solution in a way which allows easy (and accurate) translation of the model from the frequency domain to the time domain for EMTP implementation. Note, for example, that the new model structure does not involve the numerical convolution [2], or the weighting functions [3, 4, 5], of previous models.

In the new model, the transmission line is modelled by discrete *R, L, C* elements, but in an entirely different way from traditional lumped-parameter models mentioned in [6]. In particular, well-known frequency-dependent effects can be taken fully into account using what the authors believe may be a minimum set of discrete (frequency-independent) circuit elements (relative to a specified bandwidth requirement, such as 20kHz).

Furthermore, the new method does not need to assume longitudinal homogeneity. It also overcomes, in a completely different way, the problem of the frequency dependence of the transformation matrices of models based on natural modes of wave propagation [7].

## II. ANALYTICAL FOUNDATION

Instead of solving (1) and (2) in the conventional way, the new approach is based on first integrating these equations as

$$V(x) = V(l) + \int_x^l Z(\tau)I(\tau)d\tau \tag{5}$$

$$I(x) = I(l) + \int_x^l Y(\tau)V(\tau)d\tau \tag{6}$$

where $l$ is the length of the transmission line.
Note that this step does not require the introduction of any assumption regarding longitudinal homogeneity.

Whilst (5) and (6) are mathematically equivalent to (1) and (2), the new formulation has the important effect of introducing boundary values at the outset. Note, of course, that the sending-end voltages and currents are included in the formulation on setting $x$=0. Also, the formulation includes the system length.

By including boundary values and physical length, solution of (5) and (6) leads to a model which is very different from that obtained by the conventional approach (where the basic equations are solved without reference to length or boundary conditions).

In principle, solution of (5) and (6) proceeds by substituting (6) into (5) to eliminate current at all points along the system except at the boundaries. Unfortunately, this process leads to an integro-differential equation which is too difficult to solve analytically (which is probably why this approach was never pursued in the past). However, there is no difficulty in solving the equations on a discrete basis using numerical quadrature. Nevertheless, successful implementation of the approach, to yield a frequency-domain model which converts easily and accurately into an efficient time-domain counterpart, depends on a degree of ingenuity in organising the equations for solution.

## III. FREQUENCY-DOMAIN PROTOTYPE MODEL

Owing to the strong frequency dependence of the elements of the series impedance matrix $Z(x)$, it is necessary to formulate the transmission line model in the frequency domain in the first instance. The second step, discussed in section V, is to convert the frequency-domain prototype into a time-domain counterpart.

Let the multiconductor transmission line be represented in $K$ sections as shown in Fig. 1. Note that, at this stage, there is no approximation involved. In particular, there is no assumption that the sections are of equal length or that they are individually longitudinally homogeneous.

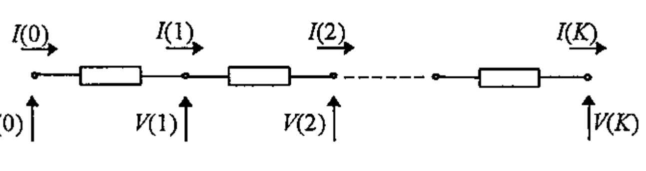


**Fig. 1 :** *One-line diagram of multiconductor transmission system split into K sections*

Each of the $K$ sections of Fig. 1 is represented by a multiterminal equivalent-$\pi$ circuit of the form shown in Fig. 2 (which shows the representation for the $k$th section).

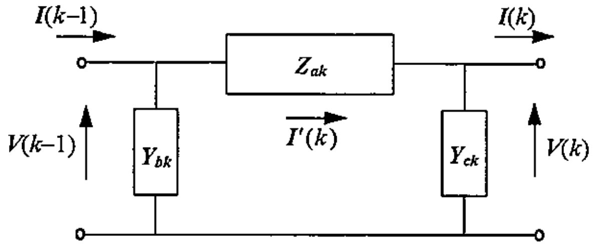


**Fig. 2 :** *Exact equivalent-$\pi$ representation of kth section*

As stated, it is not necessary for each, or any, of the sections to be longitudinally homogeneous. However, it helps clarify matters to note that if the $k$th section is of length $l_k$ and happens to be longitudinally homogeneous then the matrices in Fig. 2 would be given by

$$Z_{ak} = \sinh(\Gamma_k l_k) Z_{ok} \tag{7}$$

$$Y_{bk} = Y_{ck} = Y_{ok} \tanh\left(\frac{\Gamma_k l_k}{2}\right) \tag{8}$$

where

$$\Gamma_k = (Z_k Y_k)^{1/2}$$
$$Y_{ok} = Z_k^{-1} \Gamma_k$$

$Z_k$ = *series impedance matrix per unit length of kth section*
$Y_k$ = *shunt admittance matrix per unit length of kth section*

So far as the frequency-domain prototype is concerned, dividing the transmission line into a cascade of sections, as in Fig. 1, does not require or involve any approximation. If the line happens to be longitudinally homogeneous, then it would be logical to divide the line into sections of equal length and use (7) and (8). Indeed, the line could be represented by a single equivalent-$\pi$ section. However, this is not what is required and would not lead to the desired model structure. Nor is there any question, now or later, of simply approximating each section by an equivalent-$\pi$ circuit valid at only one frequency.

Instead, on the basis of Figs. 1 and 2, the integral equations (5) and (6) become

$$V(k) = V(K) + \sum_{j=k+1}^{K} Z_{aj} I'(j) \tag{9}$$

$$I(k) = I(K+1) + \sum_{j=k}^{K} (Y_{b,j+1} + Y_{cj})\, V(j) \tag{10}$$

Reference [8] details how these equations are solved to give a model of the structure shown in Fig. 3. The admittance equation, corresponding to this model, is

$$I_B = \{Y_b + Y''_{BB} + P_t' g P_t\} V_B \tag{11}$$


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and the voltages at discrete distances along the transmission line are given (if required) by the elements of $V'$ where

$$V' = QgP_tV_B \tag{12}$$

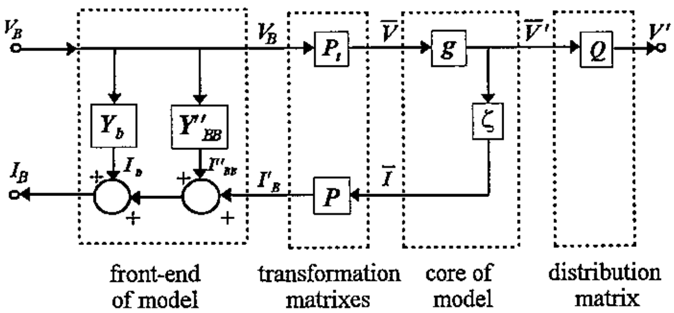


**Fig. 3 :** *New MODAL transmission-line model*

The nature of the model is revealed in the following section which relates to a simple illustrative case .

## IV. ILLUSTRATIVE LINE

Although the proposed method is applicable to all transmission lines, regardless of the number of conductors, it is sufficient for pedagogic purposes to treat the case of a single-phase line. The conductor is taken to be at a height of 15m above a semi-infinite homogeneous earth-return path of resistivity 100Ωm. The conductor itself is modelled as though it were solid aluminium (radius 1.5cm, resistivity $3.2\times10^{-8}$Ωm). A line length of 64km was chosen.

As will become clear later, the bandwidth of the eventual time-domain model depends on the number of sections into which the line is initially split. As an illustrative case, the line is split into 16 sections (each of length 4km). Since the line is longitudinally homogeneous, each section is identical. The equivalent-π representation of each section is then as shown in Fig. 4 where $Z_o = \sqrt{Z/Y}$ ; $Z$ is the series impedance of the line per unit length and $Y$ is the shunt admittance per unit length.

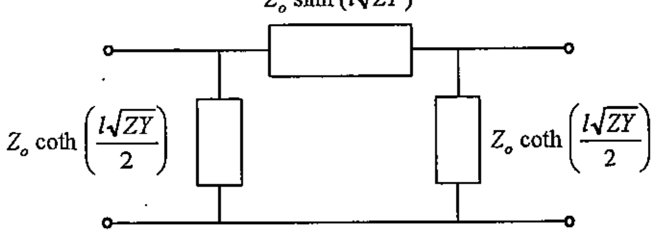


**Fig. 4 :** *Equivalent-π representation of each of the 16 sections of single-phase line (l=4km)*

The solution procedure outlined in section III converts the cascade of sections defined in Fig. 4 into a model with the structure shown in Fig. 3. $\boldsymbol{P_t}$ (the transpose of $\boldsymbol{P}$) acts as a transformation matrix. In the present case, $\boldsymbol{P_t}$ is purely real and independent of frequency as specified in the Appendix. This matrix acts on the boundary voltage vector

$$V_B = \begin{bmatrix} V_S \\ V_R \end{bmatrix}$$

(where $V_S$ and $V_R$ are the sending-end and remote-end voltages, respectively) to produce a set of 7 MODAL input voltages. The $j$th modal input voltage then acts on the $j$th element of the diagonal matrix $\boldsymbol{g}$.

The nature of the elements of $\boldsymbol{g}$ is revealed in Fig. 5. This shows the amplitude spectra of the 7 MODAL transfer functions up to a frequency of 20kHz. Each mode is seen to be characterised (within the given frequency range) by a single natural resonant frequency. The lowest natural frequency of the model is 2.10kHz which physically corresponds to fundamental natural resonance with common-mode energisation ($V_R = V_S$), i.e. corresponds to $1/2\tau$ where $\tau$ is the transit time for wave propagation from one end of the line to the other (0.24msec in this case). The second natural frequency (4.27kHz) corresponds to second-harmonic natural resonance, the third natural frequency (6.47kHz) to third-harmonic natural resonance, and so on.

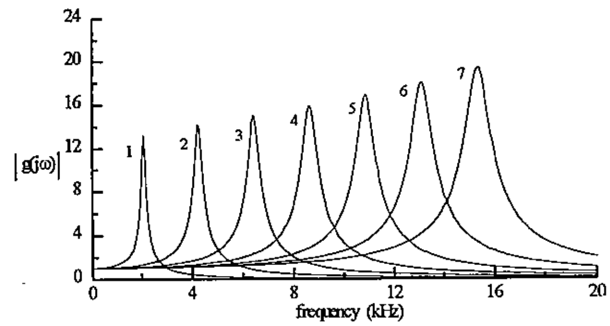


**Fig. 5 :** *Amplitude spectra of MODAL transfer functions*

This clarifies that the NEW model is centred around natural MODES of oscillation (and not around natural modes of wave propagation as in Wedepohl's model [1]). The higher-order natural resonances are less damped than the lower-order ones on account of reduced flux penetration into the ground at the higher frequencies.

The action of the elements of $\boldsymbol{g}$ (Fig. 5) is to produce a set of 7 modal output voltages (the elements of $\overline{V}'$). These are distributed along the line by the columns of $\boldsymbol{Q}$ which superposes them to give the elements of $V'$. For longitudinally homogeneous single-phase lines, $\boldsymbol{Q}$ is always purely real and completely independent of frequency. For the illustrative line, it is as given in the Appendix. The $j$th column of $Q$ acts to distribute the $j$th modal output voltage along the line. Fig. 6 illustrates these distributions graphically for the cases of modes 1-3. The columns of $Q$ are thus seen to correspond to the standing wave patterns associated with the natural resonances of the line.

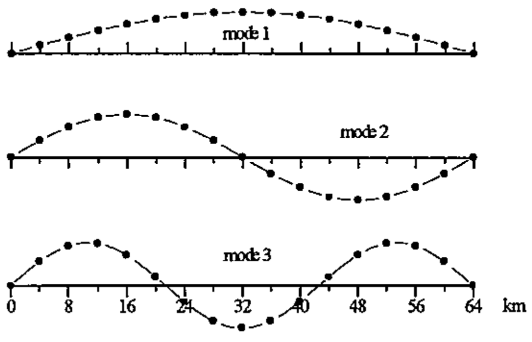


**Fig. 6 :** *Graphical representation of modal distribution vectors (modes 1-3)*

Calculations show that for frequencies up to 16kHz, the elements of the diagonal matrix $\zeta$ may be taken as purely capacitive.

$Y_b$ and $Y''_{BB}$ in the front end of the model (Fig. 3) are 2×2 matrices. The former is essentially inductive in nature (with significant frequency dependence); the latter is essentially capacitive.

The task is now to demonstrate how such a frequency-domain prototype converts to a time-domain counterpart.

## V. CONVERSION TO TIME DOMAIN

Details of the conversion for a general case (i.e. polyphase transmission lines) will be given in a forthcoming publication [9]. The present treatment illustrates basic methodology by restricting attention to the case of the single-phase line of the previous section .

The transformation matrices $P_t$ and $P$ are purely real and completely independent of frequency (given in the Appendix). The same is true of the distribution matrix $Q$ (also given in the Appendix). It follows that their action applies equally to the time domain as to the frequency domain.

The action of the elements of the diagonal matrices $g$ and $\zeta$ is now approximated by the action of *RLC* circuits of the form shown in Fig. 7.

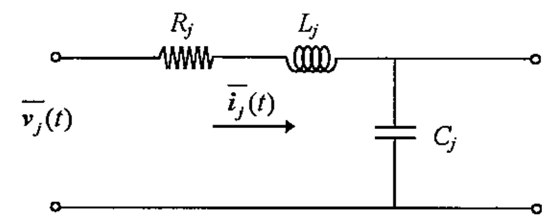


**Fig. 7 :** *RLC approximation of action of $\zeta_j$ and $g_j$*

Using the circuit values given in Table 1, Fig. 8 shows that these circuits are able to almost exactly replicate the exact amplitude spectra of Fig. 5. However, this is ONLY true at the lower frequencies. Fig. 9 compares the exact spectrum with the approximate spectrum over an extended frequency range (0-100kHz) for the particular case of mode 4 (this case is typical of the results for all the modes).

**Table 1 : Modal parameters for 64km single-phase line**

| $j$ | $f_r$ (kHz) | $C_j$ (μF) | $L_j$(H) | $R_j$ (Ω) |
|---|---|---|---|---|
| 1 | 2.1000 | 0.0293 | 0.1956 | 196.4043 |
| 2 | 4.2725 | 0.0293 | 0.0473 | 89.9357 |
| 3 | 6.4708 | 0.0293 | 0.0206 | 55.9963 |
| 4 | 8.6836 | 0.0293 | 0.0115 | 39.3908 |
| 5 | 10.9099 | 0.0293 | 0.0073 | 29.5085 |
| 6 | 13.1447 | 0.0293 | 0.0050 | 22.9221 |
| 7 | 15.3865 | 0.0293 | 0.0036 | 18.1817 |

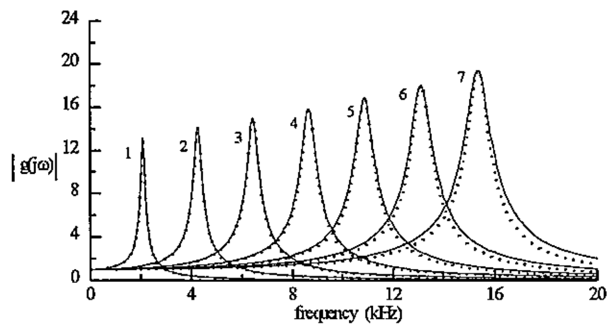


**Fig. 8 :** *Approximated spectra of MODAL transfer functions*

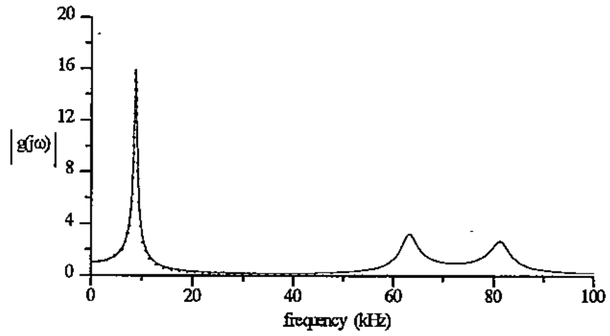


**Fig. 9 :** *Comparison between exact spectrum and approximate spectrum for mode 4*

However, by choosing to include the natural resonances shown in Fig. 5, the model is limited to a bandwidth of about 16kHz (i.e. natural resonances above this frequency are not included in the model). Thus, approximating the exact spectrum by the dotted curve in Fig. 9 only suppresses features which are outside the intrinsic bandwidth of the model.

Excluding the front-end of the model in Fig. 3, the NEW model models the transmission line in the given illustrative case using just 7 circuits of the type shown in Fig. 7 (with data as given in Table 1). Such circuits are, of course, easily and efficiently modelled using standard EMTP methodology. To ensure accurate modelling up to 16kHz, it is appropriate to choose $\Delta t$=8μsec (i.e. Shannon's theorem applied two octaves above the maximum resonant frequency).

The individual modal circuits are accessed by the action of the frequency-independent transformation matrix $P_t$ (given in the Appendix). The currents at the boundary terminals of the model are the MODAL circuit currents aggregated (see Fig. 3) by the action of $P$.

Thus, excluding the front end of the model, the given transmission line has been modelled using just 7 resistors, 7 inductors and 7 capacitors. Unfortunately, space does not permit a description of the modelling of the front end of the model (details are given in reference [9]). It is sufficient to note that modelling the front end would typically account for less than 10% of the total computational effort.

## VI. RESULTS

Fig. 10 shows the result computed by the time-domain model (with $\Delta t$=8μsec) for the case of energisation of the 64km line at crest voltage from an infinite 50Hz source. It is seen to be in remarkably good agreement with that computed using Wedepohl's full frequency-domain model. The high-frequency oscillations are due to having neglected resonant effects above 16kHz and may be digitally suppressed (if desired) to give the waveshape shown in Fig. 11. Fig. 12 shows the computed result half-way down the line (again compared with the result obtained by full-frequency-domain analysis).

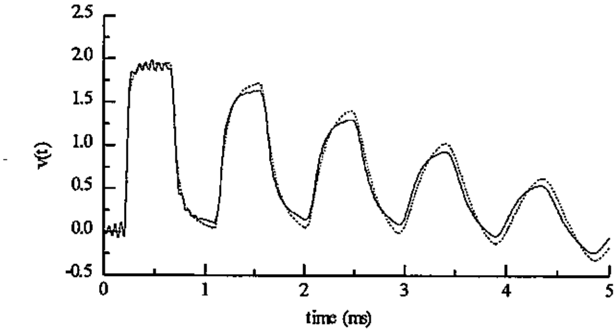


**Fig. 10 :** *Results at the receiving end of the line*
······ *Wedepohl result*
—— *Time-domain result*

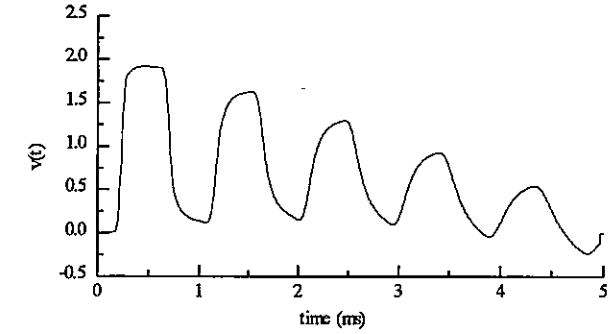


**Fig. 11 :** *Filtered version of the time-domain output*

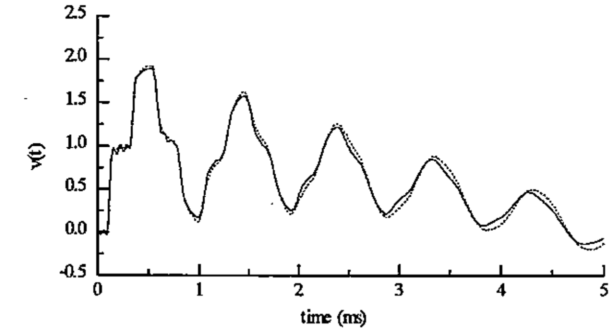


**Fig. 12 :** *Results half-way down the line*
······ *Wedepohl result*
—— *Time-domain result*

## VII. CONCLUSIONS

The paper has shown that it is possible to model transmission lines in an entirely different way from conventional transmission-line modelling and still achieve comparable results, inclusive of proper account of frequency-dependent effects.

The model is structured around natural resonances which can be modelled by simple *RLC* circuits for EMTP implementation.

The method is similar to Wedepohl's in the sense that MODAL decomposition is involved in both cases. In Wedepohl's model the decomposition is into natural MODES of wave propagation, in the proposed model the decomposition is into natural MODES of oscillation.

The quality of the computed time-domain results (Figs. 10-12) has been shown to be good notwithstanding the fact that the dominant natural frequency of these oscillations is around 1kHz whereas this frequency does not feature as a natural frequency of the model (see Table 1).

The modest errors apparent in the results given in Figs. 10 and 12 are entirely due to inaccuracies in fitting the exact spectra of Fig. 5 with the specified *RLC* approximations. At the expense of more sophisticated circuit approximations, these errors can be more or less entirely eliminated.

An important feature of the new model is that it's bandwidth is explicit (16kHz in the given example). This is important in the context of general EMTP studies where, ideally, all plant should be modelled to a same bandwidth.

The method is applicable to 3-ph lines. The computational requirement is simply tripled (i.e. there are three times as many natural resonances to be taken into account). It is important to emphasise that as computational requirements are directly proportional to the number of conductors, there is no advantage to be gained by incorporating Wedepohl-type modal decomposition.

## VIII. REFERENCES


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## IX. APPENDIX

*Modal transformation matrix* $\boldsymbol{P_t}$ :

$$P_t = \begin{bmatrix} 1.7948 & 1.7948 \\ 0.8887 & -0.8887 \\ 0.5828 & 0.5828 \\ 0.4268 & -0.4268 \\ 0.3307 & 0.3307 \\ 0.2646 & -0.2646 \\ 0.2154 & 0.2154 \end{bmatrix}$$

*Inverse modal transformation matrix* $\boldsymbol{P}$ :

$$P = \begin{bmatrix} 1.7948 & 0.8887 & 0.5828 & 0.4268 & 03307 & 0.2646 & 0.2154 \\ 1.7948 & -0.8887 & 0.5828 & -0.4268 & 0.3307 & -0.2646 & 0.2154 \end{bmatrix}$$

*Modal distribution matrix* $\boldsymbol{Q}$ :

$$Q = \begin{bmatrix} 0.0690 & 0.1353 & 0.1964 & 0.2500 & 0.2940 & 0.3266 & 0.3468 \\ 0.1353 & 0.2500 & 0.3266 & 0.3536 & 0.3266 & 0.2500 & 0.1353 \\ 0.1964 & 0.3266 & 0.3468 & 0.2500 & 0.0690 & -0.1353 & -0.2940 \\ 0.2500 & 0.3536 & 0.2500 & 0.0000 & -0.2500 & -0.3536 & -0.2500 \\ 0.2940 & 0.3266 & 0.0690 & -0.2500 & -0.3468 & -0.1353 & 0.1964 \\ 0.3266 & 0.2500 & -0.1353 & -0.3536 & -0.1353 & 0.2500 & 0.3266 \\ 0.3468 & 0.1353 & -0.2940 & -0.2500 & 0.1964 & 0.3266 & -0.0690 \\ 0.3536 & 0.0000 & -0.3536 & 0.0000 & 0.3536 & 0.0000 & -0.3536 \\ 0.3468 & -0.1353 & -0.2940 & 0.2500 & 0.1964 & -0.3266 & -0.0690 \\ 0.3266 & -0.2500 & -0.1353 & 0.3536 & -0.1353 & -0.2500 & 0.3266 \\ 0.2940 & -0.3266 & 0.0690 & 0.2500 & -0.3468 & 0.1353 & 0.1964 \\ 0.2500 & -0.3536 & 0.2500 & 0.0000 & -0.2500 & 0.3536 & -0.2500 \\ 0.1964 & -0.3266 & 0.3468 & -0.2500 & 0.0690 & 0.1353 & -0.2940 \\ 0.1353 & -0.2500 & 0.3266 & -0.3536 & 0.3266 & -0.2500 & 0.1353 \\ 0.0690 & -0.1353 & 0.1964 & -0.2500 & 0.2940 & -0.3266 & 0.3468 \end{bmatrix}$$